SyDe312 (Winter 2005)

Unit 2 - Solutions

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Chapter 3 - Root Finding for Nonlinear Equations

3.3 - 1 Secant method

Next iterate is calculated using:

$$x_{k+1} \approx x_k - f(x_k) \left[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \right]$$

In all cases the interval $[x_o, x_1] = [0, 2]$.

3.3 - 1a
$x^3 - x^2 - x - 1 = 0$

The real root is required.

We try an initial iterate $x_o = 2.0$. The application of the secant method results in the following iterations:

k	x_k	$f(x_k)$	error
0	0.0	-1.0	$-$
1	2.000000	1.000000	-1.000000
2	2.000000	-2.000000	$6.667E-01$
3	1.6666667	-0.81481	$4.583E-01$
4	2.1250000	1.9551	$-3.235E-01$
5	1.801500	-0.200340	$3.007E-01$
6	1.831600	-0.041982	$7.972E-03$
7	1.839500	0.0013563	$-2.495E-04$
8	1.8392682	$-8.6E-06$	$1.588E-06$

3.3 - 1b
$x - 1 - 0.3\cos(x) = 0$

Iterations to converge = 5
Root = 1.1284251

3.3 - 1c
$\cos x = (1/2) + \sin x$

Smallest positive root is required.

Iterations to converge = 4
Root = 0.4240310

3.3 - 1d
$x = e^{-x}$

Iterations to converge = 6
Root = 0.5671433

3.3 - 1e
$e^{-x} = \sin x$

Iterations to converge = 9
Root = 0.5885327

3.3 - 1f
$x^3 - 2x - 2 = 0$

The real root is required.

Iterations to converge = 3
Root = 1.1284251

3.3 - 1g
$x^4 - x - 1 = 0$

All reall roots are required.

Iterations to converge = 4
Root = 1.22074408

3.3 - 5 Secant method
$x^3 - 3x^2 + 3x - 1$

The final roots depend on the initial guess. The results with various initial guess are summarized below:

$[x_o, x_1]$	Root	Iterations
$[0.5, 2.0]$	0.9995659	30
$[0.5, 0.9]$	0.9970743	13
$[0.9, 1.02]$	1.0054476	6

3.3 - 6 Secant Method
$x^4 - 5.4x^3 + 10.56x^2 - 8.954x + 2.7951$

The final roots depend on the initial guess. Look for the root α in $[1, 1.2]$. The results with various initial guess are summarized below. Notice that the multiplicity of roots at $\alpha = 1.1$ causes problems when $f(x_n)$ and $f(x_{n+1})$ are too close (marked by * in the following summary).

$[x_o, x_1]$	*Root*	*Iterations*
$[1.0, 1.2]$	1.1095399	$*$
$[0.9, 1.2]$	1.1058519	$*$
$[0.0, 2.0]$	2.1000009	9
$[1.0, 2.0]$	1.0909501	$*$
$[2.0, 10.0]$	2.0999999	8

3.5 - 1 Ill-behaved Newton's Method
$p(x) = x^5 + 0.9x^4 - 1.62x^3 - 1.458x^2 + 0.6561x + 0.59049$

We use $\epsilon = 10^{-8}$ and initial guesses of -1 and 1.

Initial guess $x_o = -1$

	$x_0 = -1.0$	
k	x_k	Ratio
1	-0.9677970	
2	-0.9457323	0.6851
3	-0.9307379	0.6795
4	-0.9206112	0.6754
5	-0.9138051	0.6721
6	-0.9092333	0.6717
7	-0.9063234	0.6365
8	-0.9044989	0.6270
9	-0.9032919	0.6615
10	-0.9021632	0.9351
11	-0.9021632	0.0000

In about 11 iterations, it converged to -0.9021632. It can be seen that the ratio largely stayed around $2/3 = (m-1)/m$ pointing to multiplicity of roots to be $m = 3$.

Now, considering $x = -0.9$ a root of the given polynomial, we synthetically divide the given polynomial by $x + 0.9$ (alternatively, long division). This can be accomplished using matlab `deconv` command.

$$p_4(x) = x^4 - 1.62x^2 + 0.6561$$

Since the remainder is zero, we can say that $x = -0.9$ is a root of the given polynomial. Building on our insight gained from the ratio in the iterations of Newton's method (i.e. $m = 3$), we further synthetically divide this polynomial by $x + 0.9$ to get the following deflated polynomial:

$$p_3(x) = x^3 - 0.9x^2 - 0.81x - 0.729$$

We may once again synthetically divide the resultant polynomial by $x + 0.9$ to show that $x = -0.9$ is also a root of $p_3(x)$. Once again, a zero remainder points that $x + 0.9$ is a root for the deflated polynomial.

$$p_2(x) = x^2 - 0.81$$

(An alternate and frequently more accurate method would be to use exact/analytical formula for finding the roots of a cubic equation).

At this point it is desirable to use a (careful) quadratic formula to find the remaining roots exactly.

Initial guess $x_o = 1$

Results of Newton's method are summarized below:

	$x_0 = 1.0$	
k	x_k	Ratio
1	0.9536586	
2	0.9279458	0.5549
3	0.9142861	0.5312
4	0.9072263	0.5168
5	0.9036348	0.5087
6	0.9018209	0.5051
7	0.9009129	0.5005
8	0.9004606	0.4982
9	0.9002500	0.4655
10	0.9001275	0.5819
11	0.9000874	0.3268
12	0.9000290	1.4583
13	0.9002051	−3.015
14	0.9001305	−0.424
15	0.9000913	0.5243
16	0.8999794	2.8584
17	0.9002275	−2.217
18	0.9001377	−0.362
19	0.9000635	0.8261
20	0.9000635	0.0000

In about 20 iterations, it converged to 0.9000635. It can be seen that the ratio largely stayed around $1/2 = (m-1)/m$ pointing to multiplicity of roots to be $m = 2$.

Now, considering $x = 0.9$ a root of the given polynomial, we synthetically divide the given polynomial by $x - 0.9$ (alternatively, long division or matlab `deconv`).

$$p_4(x) = x^4 + 1.8x^3 - 1.458x - 1.9683$$

This deflation process may be repeated to get a cubic (and, subsequently, quadratic) polynomial for which we have exact analytical formulae.

3.5 - 2 Ill-behaved Newton's method
$p(x) = x^4 - 3.2x^3 + 0.96x^2 4.608x - 3.456$

We try different initial iterates. Results of Newton's method are summarized below:

$x_0 = -1.0$

k	x_k	Ratio
1	-1.2661290	
2	-1.2045493	-0.2314
3	-1.2000235	0.0734
4	-1.2000000	0.0052
5	-1.2000000	0.0000

In about 5 iterations, it converged to -1.2. It can be seen that the ratio largely stayed around $0 = (m-1)/m$ pointing to multiplicity of roots to be $m = 1$.

$x_0 = 1.0$

k	x_k	Ratio
1	2.0201614	
2	2.0010591	0.2393
3	2.0000036	0.0553
4	2.0000006	0.0029
5	2.0000002	0.0769
6	2.0000002	0.0000

In about 6 iterations, it converged to 2.0. It can be seen that the ratio largely stayed much smaller than 0.5 pointing to multiplicity of roots to be $m = 1$.

$x_0 = 1.0$

k	x_k	Ratio
1	1.0948280	
2	1.1456281	0.5357
3	1.1722769	0.5255
4	1.1859957	0.5148
5	1.19296	0.5076
6	1.1964922	0.5072
7	1.1982722	0.5039
8	1.1991687	0.5036
9	1.1996911	0.5827
10	1.2002941	1.1545
11	1.1998723	−0.699
12	1.2008450	−2.305
13	1.2003301	−0.529
14	1.999540	0.7305
15	1.2026640	−0.500
16	1.2013078	0.5259
17	1.2005945	0.7324
18	1.2000722	3.2891
19	1.1983540	−0.482
20	1.1991823	0.5497
21	1.1996375	0.5437
22	1.1998088	0.3763
23	1.2001337	1.8963
24	1.2001337	0.0000

In about 24 iterations it converged to 1.2001337. It can be seen that the ratio largely stayed around $0.5 = (m-1)/m$ pointing to multiplicity of roots to be $m = 2$. With this insight, may look at the the root of $p'(x) = 4x^3 - 9.6x^2 + 1.92x + 4.608$

With the approximate root used as the initial iterate ($x_0 = 1.2001337$), we find that it quickly converges to a root of 1.2, as shown in the following summary:

$x_0 = 1.2001337$

k	x_k	Ratio
1	1.1999999999	
2	1.1999999999	0.0000

We may also use the approximate root of 1.2 to deflate the given polynomial through synthetic division by $(x - 1.2)$ to get the following cubic polynomial: $p_3(x) = x^3 + 2x^2 + 3.36x + 8.64$

3.5 - 8 Ill-behaved Newton's method

We used the given table to find the ratios $((x_{n+1} - x_n)/(x_n - x_{n-1}))$. Results are summarized below:

n	x_n	$x_n - x_{n-1}$	Ratio
0	0.75		
1	0.752710	0.00271	
2	0.754795	0.00208	0.7675
3	0.756368	0.00157	0.7548
4	0.757552	0.00118	0.7516
5	0.758441	0.000889	0.7534

It can be seen that the ratio largely stayed around $0.75 = (m-1)/m$ pointing to multiplicity of roots to be $m = 4$. In order to find the root accurately we may use Newton's method to solve $f^{(3)}(x) = 0$ and use $x = 0.758441$ as the initial guess. Alternatively, we may try to deflate the given polynomial $f(x) = 0$ through synthetic division by $x - 0.75$. This deflation may be done repeatedly till we have deflated the polynomial for application of analytical cubic and quadratic equation formulae.

Chapter 7.3 - Nonlinear systems

7.3 - 2 Newton-Rhapson

7.3 - 2a

$(x, y) = (\pm 1.583333333333333, \pm 1.225000000000000)$

7.3 - 2b

$(x, y) = (1.770168921750883, 0.465430442834188)$
and
$(x, y) = (-1.44115096827044, 0.693376500656804)$

7.3 - 2c

$(x, y) = (0.49505850685041, 0.868859640441128)$
and
$(x, y) = (-0.847105381160620, -0.531424945978942)$

7.3 - 2d

$(x, y) = (0.215760915631622, -0.379541251533151)$
and
$(x, y) = (0.390979883845692, -1.793154775639278)$

7.3 - 3 Newton-Rhapson

Following is the summary of results obtained from different initial iterates (accuracy = $\left\|x^{(k-1)} - x^{(k)}\right\| \leq 10^{-12}$).

(x_o, y_o)	Final(x_n, y_n)	Iterations
$(1.2, 2.5)$	$(1.336355377217167, 1.754235197651699)$	6
$(-2.0, 2.5)$	$(-0.901266190783034, -2.086587594656979)$	10
$(-1.2, -2.5)$	$(-0.901266190783034, -2.086587594656979)$	6
$(2.0, -2.5)$	$(-3.001624886676722, 0.148107994958366)$	20
$(2.98, 0.15)$	$(2.998365348111602, 0.148430977729681)$	4

7.3 - 5 Newton-Rhapson

Choosing $x^{(0)} = (1, -1)$, following is the summary of iterations obtained:

k	$\left\|\alpha - x^{(k-1)}\right\|$	Ratio
0	$3.74E-01$	
21	$8.34E-02$	0.223
2	$4.13E-02$	0.495
3	$1.84E-02$	0.445
4	$8.66E-03$	0.472
5	$4.05E-03$	0.467
6	$1.82E-03$	0.450
7	$8.19E-04$	0.450
8	$3.66E-04$	0.447
9	$1.63E-04$	0.447
10	$7.29E-05$	0.446
11	$3.25E-05$	0.446
12	$1.45E-05$	0.446
13	$6.46E-06$	0.446
14	$2.88E-06$	0.446
15	$1.28E-06$	0.446
16	$5.72E-07$	0.446
17	$2.55E-07$	0.446
18	$1.14E-07$	0.446
19	$5.06E-08$	0.446
20	$2.26E-08$	0.446
21	$1.01E-08$	0.446
22	$4.48E-09$	0.446

$x^{(22)} = (1.203166968, -1.374080530)$